Deformations of -Fano threefolds and their elephants

Sano, Taro

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DEFORMATIONS OF $\mathbb{Q}$-FANO THREEFOLDS AND THEIR ELEPHANTS

TARO SANO

Contents

1. Introduction

I consider varieties over the complex number field $\mathbb{C}$ in this report. The main object in this report is a $\mathbb{Q}$-*Fano* 3-*fold*, that is, a normal projective 3-fold with only terminal singularities whose anticanonical divisor is an ample $\mathbb{Q}$-Cartier divisor. In this report, I explain my results on deformations of a $\mathbb{Q}$-Fano 3-fold. The detail is written in [San12], [San14b] and [San14a].

Since a $\mathbb{Q}$-Fano 3-fold is one of end products of the MMP, the classification of $\mathbb{Q}$-Fano 3-folds is a natural question. The classification of smooth Fano 3-folds is done by several people, including Fano, Iskovskikh and Mori-Mukai. Furthermore, Mukai classified indecomposable Fano 3-folds with canonical Gorenstein singularities and Namikawa proved that a Fano 3-fold with terminal Gorenstein singularities can be deformed to a smooth Fano 3-fold. Thus a Fano 3-fold with only terminal Gorenstein singularities can be regarded as a treatable case.

In general, a terminal singularity is not Gorenstein and the classification of non-Gorenstein $\mathbb{Q}$-Fano 3-folds is complicated and far from completion. However, a 3-fold terminal singularity can be described explicitly. In fact, it is a quotient of a compound Du Val singularity by a cyclic group action and can be deformed to several cyclic quotient singularities. Toward the classification of $\mathbb{Q}$-Fano 3-folds, Altınok-Brown-Reid ([ABR02]) conjectured the following.

Conjecture 1.1. *A $\mathbb{Q}$-Fano 3-fold has a $\mathbb{Q}$-smoothing, that is, it can be deformed to a $\mathbb{Q}$-Fano 3-fold with only cyclic quotient singularities.*

If this conjecture holds, we can reduce the classification of $\mathbb{Q}$-Fano 3-folds to those with only quotient singularities. The assumption on quotient singularities makes the classification easier as in [Tak06]. I proved the following which solves Conjecture 1.1 in most of the cases.

Theorem 1.2. *A $\mathbb{Q}$-Fano 3-fold can be deformed to a $\mathbb{Q}$-Fano 3-fold with only cyclic quotient singularities and $A_{1,2}/4$-singularities.*

Here, a $A_{1,2}/4$-*singularity* is a singularity analytically isomorphic to a terminal singularity $0 \in (x^2 + y^2 + z^3 + u^2 = 0)/\mathbb{Z}_4 \subset \mathbb{C}^4/\mathbb{Z}_4(1,3,2,1)$.

To find a deformation as in Theorem 1.2, we study infinitesimal deformations of a $\mathbb{Q}$-Fano 3-fold over Artinian rings. Thus it is important to study obstructions of infinitesimal deformations of $\mathbb{Q}$-Fano 3-folds. Locally, deformations of a 3-fold terminal singularity are unobstructed since they are induced by equivariant deformations of a compound Du Val singularity. For a $\mathbb{Q}$-Fano 3-fold, we also have the global unobstructedness as follows.

Theorem 1.3. *Deformations of a $\mathbb{Q}$-Fano 3-fold are unobstructed.*

By this theorem, the problem to find a $\mathbb{Q}$-smoothing is reduced to first order level. By computing a certain local cohomology map, we can achieve this (Lemma 4.2).

We call a member $D \in |-K_X|$ an *elephant*. A Fano 3-fold with terminal Gorenstein singularities has an elephant with only Du Val singularities and it plays an important role in the classification. However a $\mathbb{Q}$-Fano 3-fold does not necessarily have a Du Val elephant. In fact, there is an example of a $\mathbb{Q}$-Fano 3-fold X such that $|-K_X| = \emptyset$. There are also examples of $\mathbb{Q}$-Fano 3-folds with only non-Du Val elephants. Nevertheless, Altınok-Brown-Reid also conjectured the following.

Conjecture 1.4. *Let X be a $\mathbb{Q}$-Fano 3-fold. Assume there exists $D \in |-K_X|$ which is possibly very singular.*

Then there exists a deformation $\varphi\colon (\mathcal{X}, \mathcal{D}) \to \Delta^1$ of (X, D) such that $\mathcal{X}_t$ has only quotient singularities and $\mathcal{D}_t \in |-K_{\mathcal{X}_t}|$ has at most Du Val singularities only on the singularities of $\mathcal{X}_t$ for $t \neq 0$. (Such a deformation φ is called a **simultaneous $\mathbb{Q}$-smoothing***.)*

If this conjecture holds, it would be useful for the classification since we can use the rich theory of K3 surfaces. If there is an elephant with only isolated singularities, we can solve Conjecture 1.1 as follows.

Theorem 1.5. *(*[San14b, Theorem 1.1]*) Let X be a $\mathbb{Q}$-Fano 3-fold. Assume that there exists $D \in |-K_X|$ with only isolated singularities.*

Then (X, D) has a simultaneous $\mathbb{Q}$-smoothing. In particular, X has a $\mathbb{Q}$-smoothing.

There is also an example of a $\mathbb{Q}$-Fano 3-fold with only non-normal elephants. On the other hand, it is expected that, if $h^0(X, -K_X)$ is sufficiently large, X has a Du Val elephant.

2. Preliminaries

2.1. **Deformation functors.** Let $(\mathrm{Art})_{\mathbb{C}}$ be the category of local Artinian $\mathbb{C}$-algebras whose residue field $A/\mathfrak{m}_A \simeq \mathbb{C}$. Let X be an algebraic scheme and D its closed subscheme. Let $\mathrm{Def}_{(X,D)}\colon (\mathrm{Art})_{\mathbb{C}} \to (Sets)$ be a functor such that, for $A \in (\mathrm{Art})_{\mathbb{C}}$, we associate $\mathrm{Def}_{(X,D)}(A)$

the equivalence classes of deformations $(\mathcal{X}, \mathcal{D}) \to \operatorname{Spec} A$ of the pair (X, D) (cf. [San12, Definition 2.2]). When $D = \emptyset$, we set $\mathrm{Def}_X := \mathrm{Def}_{(X,\emptyset)}$. In this report, we mainly study Def_X and $\mathrm{Def}_{(X,D)}$ for a $\mathbb{Q}$-Fano 3-fold X and $D \in |-K_X|$.

A resolution of a variety with rational singularities induces the "blow-down morphism" as follows.

Proposition 2.1. *Let X be a normal variety with only rational singularities and $\mu\colon \tilde{X} \to X$ a resolution of singularities of X. Then we have a blow-down morphism*

$$\mu_*\colon \mathrm{Def}_{\tilde{X}} \to \mathrm{Def}_X$$

which sends a deformation $\tilde{\mathcal{X}} \to \operatorname{Spec} A$ of $\tilde{X}$ to $\mathcal{X} := (X, \mu_* \mathcal{O}_{\tilde{\mathcal{X}}})$.

This plays an important role in the study of deformations of rational singularities. The image of the forgetful morphism $\mathrm{Def}_{(\tilde{X},E)} \to \mathrm{Def}_{\tilde{X}}$ is called an "equisingular deformations"(cf. [Wah76, (2.4)]). In our case, we use this functor to check whether a deformation of a 3-fold terminal singularity changes the singularity.

Example 2.2. Let $0 \in S := (f = 0) \subset \mathbb{C}^3$ be a rational double point and $\mu\colon \tilde{S} \to S$ its minimal resolution. It is well-known that $\mu_*\colon \mathrm{Def}_{\tilde{S}} \to \mathrm{Def}_S$ is a finite Galois covering and it induces a zero map on the tangent spaces. Moreover, we see that $\mathrm{Def}_{(\tilde{S},E)}(A)$ only contain a trivial deformation for all $A \in (\mathrm{Art})_{\mathbb{C}}$.

Altmann determined the equisingular deformations of a 2-dimensional hypersurface singularities ([Alt87]).

2.2. **Terminal singularities.** As explained in the introduction, we have the following description of a 3-fold terminal singularity.

Let $p \in U$ be a Stein neighborhood of a 3-fold terminal singularity p of index r. It is known that $\operatorname{Sing} U \subset U$ has codimension 3, thus p is an isolated singularity. An isomorphism $\mathcal{O}_U(rK_U) \simeq \mathcal{O}_U$ induces a finite morphism $\pi_U\colon V \to U$ such that π_U is étale outside p and $K_V = \pi_U^* K_U$ is Cartier. This π_U is called an *index one cover* of U and plays an important role in the classification of 3-fold terminal singularities. By this, we see that U is a quotient of a terminal Gorenstein singularity.

It is known that a 3-fold terminal Gorenstein singularity is an isolated compound Du Val (=cDV) singularity, that is, an isolated hypersurface singularity $(f = 0) \subset \mathbb{C}^4$ defined by a polynomial $f \in \mathbb{C}[x, y, z, u]$ such that

$$f = g(x, y, z) + uh(x, y, z, u),$$

where $g \in \mathbb{C}[x, y, z]$ is a defining equation of a Du Val singularity and $h \in \mathbb{C}[x, y, z, u]$ is some polynomial. Hence a germ of a 3-fold terminal singularity (U, p) can be written as

$$(U, p) \simeq (f = 0)/\mathbb{Z}_r \subset \mathbb{C}^4/\mathbb{Z}_r(a, b, c, d),$$

where $\mathbb{C}^4/\mathbb{Z}_r(a, b, c, d)$ is a quotient of $\mathbb{C}^4$ by the action of the cyclic group $\mathbb{Z}_r$ of weights (a, b, c, d) and $f \in \mathcal{O}_{\mathbb{C}^4,0}$ is a $\mathbb{Z}_r$-semi-invariant function. Although a quotient of a cDV singularity is not necessarily terminal, we have the classification of 3-fold terminal singularities (cf. [Rei87, (3.2) Theorem]).

As a consequence of the above classification, we obtain the following important properties of 3-fold terminal singularities.

Fact 2.3. *Let $(U, p) \simeq (f = 0)/\mathbb{Z}_r$ be a germ of a 3-fold terminal singularity described as above. Then we have the following;*

(i) $\sigma \cdot f = \pm f$ *for a generator* $\sigma \in \mathbb{Z}_r$.
(ii) *A general element* $D \in |-K_U|$ *has only Du Val singularities.*
(iii) *There exists a deformation* $\phi \colon \mathcal{U} \to \Delta^1$ *of* U *over an unit disk* Δ^1 *such that* $\mathcal{U}_t$ *has only quotient singularities. (*$\mathbb{Q}$-smoothing of U*)*

Remark 2.4. If $\sigma \cdot f = f$ (resp. $\sigma \cdot f = -f$) in (i), the terminal singularity (U, p) is called *ordinary* (resp. *non-ordinary*).

Example 2.5. Let $U := (xy + h(z, u^r) = 0)/\mathbb{Z}_r \subset \mathbb{C}^4/\mathbb{Z}_r(a, -a, 0, 1)$ for coprime positive integers $a < r$. This is an example of an ordinary terminal singularity.

Let $\mathcal{U} := ((xy + h(z, u^r) + t = 0)/\mathbb{Z}_r \subset \mathbb{C}^5/\mathbb{Z}_r(a, -a, 0, 1, 0) \to \mathbb{C}$ be a deformation which sends (x, y, z, u, t) to $t \in \mathbb{C}$. We see that $\mathcal{U}$ is a $\mathbb{Q}$-smoothing of U and a general fiber has r quotient singularities.

Example 2.6. Let $U := (x^2 + y^2 + h(z, u^2) = 0)/\mathbb{Z}_4 \subset \mathbb{C}^4/\mathbb{Z}_4(1, 3, 2, 1)$ for some $h(z, u^2) \in \mathfrak{m}^2_{\mathbb{C}^4,0}$. It is known that a non-ordinary terminal singularity is of this form.

Let $\mathcal{U} := (x^2 + y^2 + h(z, u^2) = t \cdot z)/\mathbb{Z}_4 \subset \mathbb{C}^5/\mathbb{Z}_4(1, 3, 2, 1, 0)$ and $\pi \colon \mathcal{U} \to \mathbb{C}$ be the projection sending (x, y, z, u, t) to t. Then π is a $\mathbb{Q}$-smoothing of U and a general fiber has a $1/4(1, 1, 3)$-singularity and several $1/2(1, 1, 1)$-singularities.

3. Unobstructedness

3.1. **Obstruction theory for l.c.i. schemes.** We have the following description of obstruction theory in the l.c.i. case.

Proposition 3.1. *(*[Ser06, Proposition 2.4.8], [San12, Proposition 2.6]*)*

Let X *be a reduced l.c.i. algebraic scheme over a field* k *of characteristic zero. Let* $\xi_n := (\mathcal{X} \xrightarrow{f_n} \operatorname{Spec} A_n)$ *be a deformation of* X *over* $A_n := k[t]/(t^{n+1})$.

Then we can define an obstruction class $o_{\xi_n} \in \operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X)$ *for lifting* ξ_n *over* A_{n+1}.

(Construction of the obstruction). We have an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\mathcal{X}_n} \to \mathcal{O}_{\mathcal{X}_{n-1}} \to 0.$$

Since X is l.c.i., we also have an exact sequence

$$0 \to f_n^* \Omega^1_{A_n/k} \to \Omega^1_{\mathcal{X}_n/k} \to \Omega^1_{\mathcal{X}_n/A_n} \to 0.$$

Since $f_n^* \Omega^1_{A_n/k} \simeq \mathcal{O}_{\mathcal{X}_{n-1}}$, we can combine the above two sequences to obtain an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{\mathcal{X}_n} \to \Omega^1_{\mathcal{X}_n/k} \to \Omega^1_{\mathcal{X}_n/A_n} \to 0.$$

This sequence defines an element $o_{\xi_n} \in \operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X)$. We can check that, if $o_{\xi_n} = 0$, then there exists $\xi_{n+1} \in \operatorname{Def}_X(A_{n+1})$ such that $\xi_{n+1} \otimes_{A_{n+1}} A_n \simeq \xi_n$. □

3.2. **Sketch of the proof of Theorem 1.3.** A smooth Fano manifold of arbitrary dimension has unobstructed deformations. This follows since we have the vanishing of the obstruction space

$$H^2(X, \Theta_X) \simeq H^2(X, \Omega_X^{\dim X - 1} \otimes \omega_X^{-1}) = 0$$

by the Kodaira-Akizuki-Nakano vanishing theorem, where Θ_X is the tangent sheaf.

Obstructions for a Fano 3-fold with terminal Gorenstein singularities lie in $\operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X)$. In this case, we have an isomorphism

$$\operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X) \simeq \operatorname{Ext}^2(\Omega^1_X \otimes \omega_X, \omega_X) \simeq H^1(X, \Omega^1_X \otimes \omega_X)^*.$$

Namikawa proved that $H^1(X, \Omega^1_X \otimes \omega_X) = 0$ by proving a variant of the Lefschetz hyperplane section theorem [Nam97].

However we do not have the above isomorphism when ω_X is not invertible. In [San12, Theorem 2.11], I resolved this difficulty by using an explicit description of obstruction classes. Recently, I found a new method to treat this difficulty by considering the "canonical covering stack" of a $\mathbb{Q}$-Fano 3-fold. I present the sketch of the idea.

Let $\operatorname{Sing} X =: \{p_1, \dots, p_l\}$, $p_i \in U_i$ a small affine neighborhood of p_i and $\pi_i \colon V_i \to U_i$ the index one cover for $i = 1, \dots, l$. Let $U_0 := X \setminus \operatorname{Sing} X$ and $\pi_0 \colon U_0 \to X$ the open immersion. Let $U := \coprod_{i=0}^{l} U_i$ and $\pi \colon U \to X$ the morphism such that $\pi|_{U_i} = \pi_i$ for $i = 0, \dots, l$. Let $V := U \times_X U$ and consider the étale groupoid space

$$V \overset{p_1}{\underset{p_2}{\rightrightarrows}} U.$$

Let $\mathfrak{X}$ be the associated Deligne-Mumford stack. Let $c \colon \mathfrak{X} \to X$ be the morphism to the coarse moduli space. We can define a functor $\operatorname{Def}_{\mathfrak{X}} \colon (Art)_{\mathbb{C}} \to (Sets)$ of deformations of the stack $\mathfrak{X}$ over Artinian rings as in the case of schemes. We see that there is an isomorphism of functors

$$c_* \colon \operatorname{Def}_{\mathfrak{X}} \xrightarrow{\sim} \operatorname{Def}_X \tag{1}$$

which sends a deformation of $\mathfrak{X}$ to its coarse moduli space. We can construct obstructions for deformations of $\mathfrak{X}$ similarly as Proposition 3.1.

Proposition 3.2. *Let X be a 3-fold with terminal singularities and $\mathfrak{X}$ its canonical covering stack.*

Then we can define an obstruction $o_\xi \in \operatorname{Ext}^2(\Omega^1_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}})$ for each deformation $\xi \in \operatorname{Def}_{\mathfrak{X}}(A)$ over an Artinian ring A.

Sketch of proof of Theorem 1.3. By the isomorphism (1), it is enough to show that $\operatorname{Def}_{\mathfrak{X}}$ is a smooth functor. We have isomorphisms

$$\operatorname{Ext}^2(\Omega^1_{\mathfrak{X}}, \mathcal{O}_{\mathfrak{X}}) \simeq \operatorname{Ext}^2(\Omega^1_{\mathfrak{X}} \otimes \omega_{\mathfrak{X}}, \omega_{\mathfrak{X}}) \simeq H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}} \otimes \omega_{\mathfrak{X}})^*.$$

The first isomorphism follows since $\omega_{\mathfrak{X}}$ is invertible. (This is a main advantage of considering the canonical covering stack.) The second isomorphism follows from the Serre duality on Deligne-Mumford stacks. Moreover, we have an isomorphism

$$H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}} \otimes \omega_{\mathfrak{X}}) \simeq H^1(X, \iota_*(\Omega^1_{X'} \otimes \omega_{X'})),$$

where $\iota \colon X' \hookrightarrow X$ is an open immersion of the smooth part X' of X. We can check this by the construction. Thus it is enough to check $H^1(X, \iota_*(\Omega^1_{X'} \otimes \omega_{X'})) = 0$. This can be checked by a variant of Lefschtz hyperplane section theorem as in [San12, Theorem 2.11].

□

4. $\mathbb{Q}$-smoothings

Let X be a projective variety with only isolated singularities. We have the following exact sequence induced by the spectral sequence for Ext groups;

$$\begin{aligned} 0 \to H^1(X, \Theta_X) \to \operatorname{Ext}^1(\Omega^1_X, \mathcal{O}_X) \xrightarrow{\Pi} H^0(X, \underline{\operatorname{Ext}}^1(\Omega^1_X, \mathcal{O}_X)) \\ \to H^2(X, \Theta_X) \to \operatorname{Ext}^2(\Omega^1_X, \mathcal{O}_X). \end{aligned} \tag{2}$$

Note that $\mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \simeq \mathrm{Def}_X(A_1)$ is the set of first order deformations of X and $H^0(X, \underline{\mathrm{Ext}}^1(\Omega^1_X, \mathcal{O}_X))$ is the set of first order deformations of singularities of X. Furthermore Π corresponds to restriction of global deformations to local deformations of singularities. Hence, if we have $H^2(X, \Theta_X) = 0$, we can always lift deformations of singularities to a deformation of X on the first order level, at least.

Let us consider the case where X is a $\mathbb{Q}$-Fano 3-fold. Namikawa constructed an example of a Fano 3-fold X with terminal Gorenstein singularities such that $H^2(X, \Theta_X) \neq 0$ ([Nam97, Example 5]). Thus we need a more precise argument.

The key tool to find a $\mathbb{Q}$-smoothing is a coboundary map of some local cohomology group associated with a resolution of a singularity. Let us first define the map for a Gorenstein 3-fold singularity.

Let $q \in V$ be a Stein neighborhood of an isolated 3-fold Gorenstein singularity and $\nu \colon \tilde{V} \to V$ a resolution of the singularity whose exceptional locus F is a SNC divisor such that ν induces an isomorphism over $V' := V \setminus \{q\}$. We can consider a coboundary map

$$\phi_V \colon H^1(V', \Omega^2_{V'}) \to H^2_F(\tilde{V}, \Omega^2_{\tilde{V}}(\log F)(-F))$$

Since V is Gorenstein and $q \in V$ has codimension 3, we have $H^1(V', \Omega^2_{V'}) \simeq \mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V)$. Namikawa-Steenbrink proved the following.

Theorem 4.1. *(*[NS95, Theorem 1.1]*) Let (V, q) be an isolated Gorenstein Du Bois singularity.*

Then $\phi_V = 0$ if and only if (V, q) is infinitesimally-rigid.

Moreover we have $\dim \mathrm{Ker}\, \phi_V \leq \dim \mathrm{Im}\, \phi_V$.

In particular, $\phi_V \neq 0$ for a terminal Gorenstein singularity (V, q). Namikawa-Steenbrink used this to find a smoothing of a $\mathbb{Q}$-factorial Calabi-Yau 3-fold with terminal singularities.

We can consider a similar coboundary map for a non-Gorenstein terminal singularity (U, p) of index r as follows; Let $\pi_U \colon V \to U$ be an index one cover of U and $\nu \colon \tilde{V} \to V$ a $\mathbb{Z}_r$-equivariant resolution with the same properties as above. We can consider the coboundary map ϕ_V for this resolution as above. Let ϕ_U be a coboundary map

$$\phi_U \colon H^1(U', \Omega^2_{U'}(-K_{U'})) \to H^2_E(\tilde{U}, \mathcal{F}_{\tilde{U}}),$$

where $\mathcal{F}_{\tilde{U}}$ is the $\mathbb{Z}_r$-invariant part of the sheaf $\tilde{\pi}_* \Omega^2_{\tilde{V}}(\log F)(-F - \nu^* K_V)$. Note that $\mathcal{F}_{\tilde{U}}|_{U'} \simeq \Omega^2_{U'}(-K_{U'})$.

The following is a key lemma to find a good first order deformations of a $\mathbb{Q}$-Fano 3-fold.

Lemma 4.2. *(*[San14b, Theorem 1.3]*) Let (U, p) be a 3-fold terminal singularity.*

Then $\phi_U = 0$ if and only if (U, p) is a quotient singularity or a $A_{1,2}/4$-singularity.

Sketch of Proof. We regard $T^1_U \subset T^1_V$ as the set of $\mathbb{Z}_r$-invariant elements.

For an ordinary terminal singularity (U, p), we have $\phi_U \neq 0$ since ϕ_U is an $\mathcal{O}_{U,p}$-module homomorphism and the generator $\bar{1} \in T^1_V$ is still contained in T^1_U. Thus the statement $\phi_V \neq 0$ in Theorem 4.1 directly implies $\phi_U \neq 0$.

For a non-ordinary terminal singularity (U, p), the generator $\bar{1} \in T^1_V$ is not contained in T^1_U. Nevertheless, if (U, p) is not an $A_{1,2}/4$-singularity, we can compute $\phi_U \neq 0$ by the relation $\dim \mathrm{Ker}\, \phi_V \leq \dim \mathrm{Im}\, \phi_V$. □

Sketch of (Lemma 4.2 ⇒ Theorem 1.2). Let $\mathrm{Sing}\, X := \{p_1, \ldots, p_l\}$ and $p_i \in U_i$ a Stein neighborhood of p_i for $i = 1, \ldots, l$. Let $Y \to X$ be a cyclic cover determined by a smooth member $D_m \in |-mK_X|$ for a sufficiently large integer m. Let $\nu \colon \tilde{Y} \to Y$ be a $\mathbb{Z}_m$-equivariant resolution, $F \subset \tilde{Y}$ the ν-exceptional divisor , $\tilde{\pi}\colon \tilde{Y} \to \tilde{X} := \tilde{Y}/\mathbb{Z}_m$ the quotient morphism. Let $\mu\colon \tilde{X} \to X$ be the induced birational morphism and $E \subset \tilde{X}$ the μ-exceptional divisor. Let $\tilde{U}_i := \mu^{-1}(U_i)$ and $E_i := E \cap \tilde{U}_i$. We use the following commutative diagram;

$$\begin{array}{ccccc} T^1_X & \longrightarrow & H^2_E(\tilde{X}, \mathcal{F}_{\tilde{X}}) & \longrightarrow & H^2(\tilde{X}, \mathcal{F}_{\tilde{X}}) \\ \downarrow{\scriptstyle \oplus \mathrm{pr}_{U_i}} & & \downarrow{\scriptstyle \simeq} & & \\ \bigoplus_{i=1}^l T^1_{U_i} & \xrightarrow{\oplus \phi_{U_i}} & \bigoplus_{i=1}^l H^2_{E_i}(\tilde{U}_i, \mathcal{F}_{\tilde{U}_i}), & & \end{array}$$

where $\mathcal{F}_{\tilde{X}}$ is the $\mathbb{Z}_m$-invariant part of $\tilde{\pi}_*\Omega^2_{\tilde{Y}}(\log F)(-F + \nu^*\pi^*(-K_X))$. Let $\eta_i \in T^1_{U_i}$ be an element such that $\phi_{U_i}(\eta_i) \neq 0$. Note that $H^2(\tilde{X}, \mathcal{F}_{\tilde{X}}) = 0$ since this is a direct summand of $H^2(\tilde{Y}, \Omega^2_{\tilde{Y}}(\log F)(-F) \otimes (\nu^*\pi^*(-K_X)))$ which vanishes by the Guillen-Navarro Aznar-Puerta-Steenbrink vanishing theorem. Hence we can lift $(\phi_{U_i}(\eta_i)) \in H^2_E(\tilde{X}, \mathcal{F}_{\tilde{X}})$ to an element $\eta \in T^1_X$ such that $\mathrm{pr}_{U_i}(\eta) - \eta_i \in \operatorname{Ker} \phi_{U_i}$. Since $\operatorname{Ker} \phi_{U_i} \subsetneq T^1_{U_i}$, we see that η improves the singularity p_i until we reach an $A_{1,2}/4$-singularity. In fact, we can check that η is not an equisingular direction, that is, it does not come from the resolution of U_i. For detail, see the proof of [San12, Theorem 3.5].

□

5. Deformations of a $\mathbb{Q}$-Fano 3-fold and its anticanonical element

Let X be a $\mathbb{Q}$-Fano 3-fold and $D \in |-K_X|$ an elephant with only isolated singularities. We can assume that X has only quotient singularities and $A_{1,2}/4$-singularities by Theorem 1.2. Takagi proved the following fact on the non-Du Val singularities of general elephants of a $\mathbb{Q}$-Fano 3-fold.

Theorem 5.1. *(*[Tak02, Proposition 1.1]*) Let X be a $\mathbb{Q}$-Fano 3-fold. Assume that there exists $D_0 \in |-K_X|$ which is normal at the non-Gorenstein points of X.*

Then there exists $D \in |-K_X|$ which is normal on X and Du Val outside the non-Gorenstein points of X.

Thus it is enough to consider neighborhoods $U_1, \ldots, U_l$ of the singular points $\{p_1, \ldots, p_l\}$ of X.

Although a $\mathbb{Q}$-Fano 3-fold with only quotient singularities is easier to treat, it may not have a Du Val elephant as in the following example.

Example 5.2. ([San14a, Example 4.4]) Let $X := (x^{15} + xy^7 + z^5 + w_1^3 + w_2^3 = 0) \subset \mathbb{P}(1, 2, 3, 5, 5)$ be a weighted hypersurface, where x, y, z, w_1, w_2 are coordinate functions with degrees $1, 2, 3, 5, 5$ respectively. We can check that X is a $\mathbb{Q}$-Fano 3-fold with only terminal quotient singularities. Moreover we have $|-K_X| = \{D\}$, where $D := (z^5 + w_1^3 + w_2^3 = 0) \subset \mathbb{P}(2, 3, 5, 5)$ is an elephant with a non-log canonical singularity at $[1 : 0 : 0 : 0]$.

We can construct a simultaneous $\mathbb{Q}$-smoothing of a pair (X, D) explicitly by considering a general hypersurface. For example, $\mathcal{X} := (x^{15}+xy^7+z^5+w_1^3+w_2^3 = \lambda \cdot y^6 z) \subset \mathbb{P}(1, 2, 3, 5, 5) \times \mathbb{C}_\lambda$ induces a simultaneous $\mathbb{Q}$-smoothing of (X, D).

5.1. **Necessary local results.** The strategy of the proof can be regarded as a pair version of that of Theorem 1.2. We also use the coboundary map of certain local cohomology group and the blow-down morphism of deformations. In this case, we should carefully choose a log resolution of singularities of the pair (X, D).

Let X be a normal variety with only rational singularities and D its Cartier divisor. Let $\mu\colon \tilde{X} \to X$ be a proper birational morphism from a smooth variety $\tilde{X}$ such that the support of $\mu^{-1}(D)$ is a SNC divisor. Let $\tilde{D}$ be the strict transform of D and E the μ-exceptional divisor. Then we can define a functor

$$\mu_*\colon \operatorname{Def}_{(\tilde{X}, \tilde{D}+E)} \to \operatorname{Def}_{(X,D)}$$

which we also call the "blow-down morphism" of deformations (cf. [San14a])

Let us consider our main situation. Let X be a 3-fold with only terminal singularities and D its $\mathbb{Q}$-Cartier divisor with only isolated singularities. In this case, we can define a blow-down morphism in another way. Let $\mu\colon \tilde{X} \to X$ be a proper birational morphism from another normal variety $\tilde{X}$ such that μ is an isomorphism over $X' := X \setminus \operatorname{Sing} D$. Let $\tilde{D} \subset \tilde{X}$ be the strict transform, $E \subset \tilde{X}$ the exceptional locus and $D' := D \setminus \operatorname{Sing} D$. Then we can define $\mu_*\colon \operatorname{Def}_{(\tilde{X}, \tilde{D}+E)} \to \operatorname{Def}_{(X,D)}$ as a composition

$$\operatorname{Def}_{(\tilde{X}, \tilde{D}+E)} \xrightarrow{\tilde{\iota}^*} \operatorname{Def}_{(X',D')} \xrightarrow{\simeq} \operatorname{Def}_{(X,D)},$$

where $\tilde{\iota}^*$ is a restriction by an open immersion $\tilde{\iota}\colon X' \hookrightarrow \tilde{X}$. Note that the isomorphism $\operatorname{Def}_{(X',D')} \xrightarrow{\simeq} \operatorname{Def}_{(X,D)}$ follows since the codimension of $\operatorname{Sing} D \subset X$ is 3.

The image of the blow-down morphism does not change the badness of the singularity in the following sense.

Example 5.3. Let $U := \mathbb{C}^3$ and $D := (f=0) \subset U$ a divisor with only isolated singularities. Let $\mu\colon \tilde{U} \to U$ be a log resolution of a pair (U,D). Let $T^1_{(U,D)}$ be the set of first order deformations of (U,D). Then we can write $T^1_{(U,D)} \simeq \mathcal{O}_{U,0}/J_f$, where J_f is the Jacobian ideal. Thus $T^1_{(U,D)}$ has a $\mathcal{O}_{U,0}$-module structure. In this situation, we have the relation

$$\operatorname{Im} \mu_* \subset \mathfrak{m}^2_{U,0} \cdot T^1_{(U,D)}.$$

In particular, a smoothing can not be contained in the image of the blow-down morphism μ_* on first order level.

Let U be a Stein neighborhood of a singularity of X and $D_U \in |-K_U|$ an element with a non-Du Val singularity at $0 \in D_U$. We can assume that $U = \mathbb{C}^3/\mathbb{Z}_r(1,a,r-a)$ for coprime $a<r$ or $U = (x^2+y^2+z^3+u^2=0)/\mathbb{Z}_4(1,3,2,1)$. Let $\mu_1\colon U_1 \to U$ be the weighted blow-up with weights $1/r(1,a,r-a)$ (resp. $1/4(1,3,2,1)$) when $U = \mathbb{C}^3/\mathbb{Z}_r(1,a,r-a)$ (resp. $U = (x^2+y^2+z^3+u^2=0)/\mathbb{Z}_4(1,3,2,1)$). Let $\mu_2\colon \tilde{U} \to U_1$ be a log resolution of a pair $(U_1, \mu_1^{-1}(D))$ as constructed in [San14a, Lemma3.6]. We construct a log resolution of (U,D) as a composition

$$\tilde{U} \xrightarrow{\mu_2} U_1 \xrightarrow{\mu_1} U.$$

We define the coboundary map as

$$\tau_U\colon H^1(U', \Omega^2_{U'}(\log D'_U)) \to H^2_{E_U}(\tilde{U}, \Omega^2_{\tilde{U}}(\log \tilde{D}_U + E_U)),$$

where $U' := U \setminus \{p\}$, $D'_U := D \cap U'$ and $E_U := \mu_U^{-1}(p)$.

Define the blow-down morphism as;

$$(\mu_{U,1})_*\colon T^1_{(U_1, D_{U,1}+E_{U,1})} \to T^1_{(U',D'_U)} \xrightarrow{\simeq} T^1_{(U,D_U)}.$$

We can prove the relation

$$\operatorname{Im}(\mu_{U,1})_* \subset \mathfrak{m}^2 T^1_{(U,D)} \tag{3}$$

which can be regarded as an equisingular property (See [San14a, Lemmas 3.7]).

Moreover we can prove the non-vanishing of the coboundary map by the following stronger statement.

Lemma 5.4. *(*[San14a, Lemma 3.10]*) Let $p \in U$ be a neighborhood of a 3-fold terminal singularity and $p \in D_U \in |-K_U|$ a non Du Val elephant.*

Then we have

$$\operatorname{Ker} \tau_U \subset \operatorname{Im}(\mu_{U,1})_* \subsetneq H^1(U', \Omega^2_{U'}(\log D'_U)) \simeq T^1_{(U,D_U)}.$$

5.2. **Sketch of the proof.** Let $p_1, \dots, p_l \in \operatorname{Sing} D$ be the non Du Val points of $D \in |-K_X|$ and $p_i \in U_i$ a Stein neighborhood for $i = 1, \dots, l$. Construct $\mu \colon \tilde{X} \to X$ by patching each $\mu_i \colon \tilde{U}_i \to U_i$ for $i = 1, \dots, l$. Consider the diagram

$$\begin{array}{ccccc} H^1(X', \Omega^2_{X'}(\log D')) & \longrightarrow & H^2_E(\tilde{X}, \Omega^2_{\tilde{X}}(\log \tilde{D} + E)) & \longrightarrow & H^2(\tilde{X}, \Omega^2_{\tilde{X}}(\log \tilde{D} + E)) \\ \Big\downarrow {\scriptstyle \oplus p_{U_i}} & & \Big\downarrow & & \\ \oplus_{i=1}^l H^1(U'_i, \Omega^2_{U'_i}(\log D'_i)) & \xrightarrow{\oplus \tau_{U_i}} & \oplus_{i=1}^l H^2_{E_i}(\tilde{U}_i, \Omega^2_{\tilde{U}_i}(\log \tilde{D}_i + E_i)), & & \end{array}$$

where $X' := X \setminus \operatorname{Sing} D, D' := D \cap X'$ and so on. Note that, since $\operatorname{Sing} D \subset X$ has codimension 3, we have $T^1_{(X,D)} \simeq T^1_{(X',D')}, T^1_{(U_i,D_i)} \simeq T^1_{(U'_i,D'_i)}$ and the homomorphism $\oplus p_{U_i}$ in the diagram can be regarded as a restriction homomorphism $T^1_{(X,D)} \to \oplus T^1_{(U_i,D_i)}$.

From this diagram, we obtain a global deformation $\eta \in T^1_{(X,D)}$ which induces a deformation of D to a Du Val elephant as follows. Let $\eta_i \in T^1_{(U_i,D_i)}$ be an element inducing a simultaneous $\mathbb{Q}$-smoothing of (U_i, D_i). Then $\tau_{U_i}(\eta_i) \neq 0$ by Lemma 5.4. Since we have $H^2(\tilde{X}, \Omega^2_{\tilde{X}}(\log \tilde{D} + E)) = 0$ by the affineness of the complement $\tilde{X} \setminus (\tilde{D} \cup E) \simeq X \setminus D$, we obtain $\eta \in H^1(X', \Omega^2_{X'}(\log D')) \simeq T^1_{(X,D)}$ such that $p_{U_i}(\eta) - \eta_i \in \operatorname{Ker} \tau_{U_i}$. Lemma 5.4 implies that $p_{U_i}(\eta) - \eta_i \in \operatorname{Im}(\mu_{U_i,1})_*$. By the relation (3), we see that $p_{U_i}(\eta) - \eta_i \in \mathfrak{m}^2 T^1_{(U_i,D_i)}$ and this implies that $p_{U_i}(\eta)$ also induces a simultaneous $\mathbb{Q}$-smoothing of (U_i, D_i). Thus we obtain a deformation $(\mathcal{X}, \mathcal{D}) \to \Delta^1$ such that $\mathcal{D}_t$ has only Du Val singularities.

However the general fiber $\mathcal{X}_t$ may have $A_{1,2}/4$-singularities if D has Du Val singularities on the $A_{1,2}/4$-singularities on X. We can argue similarly to deform $(\mathcal{X}_t, \mathcal{D}_t)$ to a V-smooth pair, that is, a pair locally isomorphic to $(\mathbb{C}^3/\mathbb{Z}_r(1, a, r-a), (x = 0)/\mathbb{Z}_r)$ for some coprime integers r, a (See [San12, Theorem 1.9]).

6. Applications and further problems

A $\mathbb{Q}$-Fano 3-fold X is called *primary* if its class group is generated by the anticanonical class $-K_X$ modulo torsions. It is a generalization of a smooth Fano 3-fold of index one with Picard number one and considered to be the main class in the classification of $\mathbb{Q}$-Fano 3-folds. Takagi ([Tak06, Theorem 1.5]) obtained the genus bound

$$h^0(X, -K_X) \leq 10$$

of a non-Gorenstein primary $\mathbb{Q}$-Fano 3-fold X by assuming that X has only quotient singularities and there exists a Du Val elephant on X. As an application of Theorem 1.5, we obtain the following.

Theorem 6.1. *Let X be a non-Gorenstein primary $\mathbb{Q}$-Fano 3-fold. Assume that X has an elephant with only isolated singularities.*

Then we have $h^0(X, -K_X) \leq 10$.

On the other hand, it is expected that a primary $\mathbb{Q}$-Fano 3-fold has a Du Val elephant if the genus is sufficiently large. Takagi ([Tak02, Corollary 1.2]) proved the existence of a Du Val elephant on a primary $\mathbb{Q}$-Fano 3-fold whose Gorenstein index is 2 such that $h^0(X, -K_X) \geq 4$.

It is also interesting to study which K3 surface with Du Val singularities can appear as an elephant of a $\mathbb{Q}$-Fano 3-fold. Beauville studied this problem in the smooth case([Bea04]).

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Max Planck Institute for Mathematics, 7 Vivatsgasse, Bonn, 53111, Germany